Journal of Applied Mechanics and Technical Physics, Vol. 49, No. 4, pp. 693–698, 2008

MODEL OF A STRONG DISCONTINUITY FOR THE EQUATIONS OF SPATIAL LONG WAVES PROPAGATING IN A FREE-BOUNDARY SHEAR FLOW

V. M. Teshukov* **and A. K. Khe**

UDC 533.6.011 + 517.948.34

The long-wave equations describing three-dimensional shear wave motion of a free-surface ideal fluid are rearranged to a special form and used to describe discontinuous solutions. Relations at the discontinuity front are derived, and stability conditions for the discontinuity are formulated. The problem of determining the flow parameters behind the discontinuity front from known parameters before the front and specified velocity of motion of the front are investigated.

Key words: *vortex shallow water, hydraulic jump, integrodifferential equations.*

Introduction. We consider a mathematical model which describes three-dimensional shear flows of a heavy incompressible ideal fluid with a free surface above an even bottom in the long-wave approximation. This model, which extends the classical shallow water model, reduces to a system of integrodifferential equations. Unlike for the classical model, the propagation of nonlinear wave perturbations for the integrodifferential model has been studied less extensively. New approaches to the solution of these questions were proposed in [1–4] using a new mathematical apparatus. In [1], generalized characteristics were found and hyperbolicity conditions were formulated for the system of integrodifferential shallow-water equations describing three-dimensional stationary shear flows of a free-boundary ideal fluid. The spatial simple waves described by the indicated system of equations were studied in [2]. A definition of discontinuous solutions for a mathematical model of shear plane–parallel incompressible flows was proposed in [3]. In the same paper, the properties of strong-discontinuity relations were analyzed. A similar analysis for a model of plane–parallel flows of barotropic fluids was performed in [4]. New approaches to the description of the interaction of shear flows of an incompressible ideal fluid were used in [5]. Problems of conjugation of various filtration and channel flows of a viscous incompressible fluid and various mathematical models of two-phase fluids were studied in [6, 7].

In the present paper, we consider the spatial problem of conjugation of non-one-dimensional flows of an ideal incompressible fluid with a free boundary. Relations at the discontinuity front and stability conditions for discontinuous flow are formulated.

1. Formulation of the Problem. We consider the system of equations

$$\begin{gathered} u_t + uu_x + vu_y + wu_z + p_x/\rho = 0, \qquad v_t + uv_x + vv_y + wv_z + p_y/\rho = 0, \\ p_z = -\rho g, \qquad u_x + v_y + w_z = 0, \end{gathered} \tag{1.1}$$

which describes three-dimensional ideal incompressible fluid flows in a gravity field in the long-wave approximation. System (1.1) is obtained from the exact Euler equations by asymptotic decomposition in the small parameter $\varepsilon = H_0/L_0$ (H_0 and L_0 are the characteristic vertical and horizontal scales; it is assumed that $H_0/L_0 \ll 1$). Here u, v, and w are the fluid velocity components, p is the pressure, $\rho = \text{const}$ is the density, t is time, and x, y, and z are Cartesian coordinates in space.

*Deceased.

Lavrent'ev Institute of Hydrodynamics, Siberian Division, Russian Academy of Sciences, Novosibirsk 630090; alekhe@hydro.nsc.ru. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 49, No. 4, pp. 206–213, July–August, 2008. Original article submitted August 22, 2007.

Below, we consider fluid flow in a layer with a free boundary $0 \leqslant z \leqslant h(t,x,y)$. On the free surface $z = h(t,x,y)$, the pressure is assumed to be constant: $p = p_0$. On the boundaries of the fluid layer, the kinematic conditions $w = 0$ for $z = 0$ and $w = h_t + uh_x + vh_y$ for $z = h$ should be satisfied.

At the initial time $t = 0$, the velocity field and the free-boundary shape are specified:

$$u\Big|_{t=0} = u_0(x,y,z), \quad v\Big|_{t=0} = v_0(x,y,z), \quad w\Big|_{t=0} = w_0(x,y,z), \quad h\Big|_{t=0} = h_0(x,y).$$

The free-boundary problem for system (1.1) has exact solutions defined by the relations $u_z = 0$ and $v_z = 0$. In this class, free-boundary flows are described by the classical shallow-water equations

$$u_t + uu_x + vu_y + gh_x = 0, \qquad v_t + uv_x + vv_y + gh_y = 0, \qquad h_t + (uh)_x + (vh)_y = 0.$$

We consider shear flows of the general form described by the inequality $u_z^2 + v_z^2 \neq 0$. Modeling of these flows yields more complex integrodifferential systems of equations, which will be considered below.

Free-boundary shear flows are conveniently analyzed in mixed Eulerian–Lagrangian coordinates x', y', and λ since the unknown boundary is fixed in these coordinates. Conversion to the new coordinates is performed using the formulas [2]:

$$x' = x, \qquad y' = y, \qquad \Phi(t,x',y',\lambda) = z.$$

Here the function $\Phi(t,x',y',\lambda)$ is a solution of the Cauchy problem

$$\Phi_t + u(t,x,y,\Phi)\,\Phi_x + v(t,x,y,\Phi)\,\Phi_y = w(t,x,y,\Phi), \qquad \Phi\Big|_{t=0} = \Phi_0(x,y,\lambda).$$

The function $\Phi_0(x,y,\lambda)$ is chosen so that the value $\lambda = 0$ correspond to the even bottom $[\Phi_0(x,y,0) = 0]$ and $\lambda = 1$ to the free surface $[\Phi_0(x,y,1) = h_0(x,y)]$. In the new variables, a fixed layer $0 \leqslant \lambda \leqslant 1$, corresponds to the region occupied by the fluid and the system of equations of three-dimensional flows becomes

$$\begin{gathered} u_t + uu_x + vu_y + gh_x = 0, \qquad v_t + uv_x + vv_y + gh_y = 0, \\ H_t + (uH)_x + (vH)_y = 0. \end{gathered} \tag{1.2}$$

This system serves to determine the functions $u(t,x,y,\lambda)$, $v(t,x,y,\lambda)$, and $H(t,x,y,\lambda)$. In (1.2), we introduced a new required function $H(t,x,y,\lambda) = \Phi_\lambda(t,x,y,\lambda)$, which is linked to the depth of the fluid layer h by the relation $h = \displaystyle\int_0^1 H\,d\lambda$.

2. Generalized Solutions of Long-Wave Equations. As shown by examples of plane–parallel flows, system (1.2) is characterized by breaking of smooth solutions in a finite time, which leads to the necessity of studying generalized solutions in the classes of discontinuous functions. Usually, discontinuous solutions are introduced in writing equations of motion in the form of conservation laws. However, Eqs. (1.2) cannot be written in divergent form. The problem of determining discontinuous solutions of nondivergent equations has long attracted the attention of researchers. In the present work, the discontinuous solutions are determined using the following nondivergent form of Eqs. (1.2):

$$\begin{gathered} u_{\lambda t} + \Big(\frac{u^2+v^2}{2}\Big)_{\lambda x} - (v(v_x - u_y))_\lambda = 0, \qquad v_{\lambda t} + \Big(\frac{u^2+v^2}{2}\Big)_{\lambda y} + (u(v_x - u_y))_\lambda = 0, \\ H_t + (uH)_x + (vH)_y = 0, \\ \Big(\int_0^1 Hu\,d\lambda\Big)_t + \Big(\int_0^1 Hu^2\,d\lambda + \frac{gh^2}{2}\Big)_x + \Big(\int_0^1 Huv\,d\lambda\Big)_y = 0, \\ \Big(\int_0^1 Hv\,d\lambda\Big)_t + \Big(\int_0^1 Huv\,d\lambda\Big)_x + \Big(\int_0^1 Hv^2\,d\lambda + \frac{gh^2}{2}\Big)_y = 0. \end{gathered} \tag{2.1}$$

We note that, for smooth solutions, system (2.1) is equivalent to system (1.2).

3. Relations at a Strong Discontinuity. We consider the class of generalized solutions of Eqs. (2.1) that describe flows with a strong discontinuity. We assume that the solution has a discontinuity of the first kind on a cylindrical surface Γ specified by the equation $S(t,x,y)=0$ in the space of variables (t,x,y,λ) and that, on both sides of the surface, it is continuously differentiated. In the space of variables (t,x,y), the normal vector $\boldsymbol{\xi}$ to Γ has the form

$$\boldsymbol{\xi}=(\tau,\xi,\eta)=\frac{(S_t,S_x,S_y)}{\sqrt{S_t^2+S_x^2+S_y^2}}.$$

Let $\boldsymbol{u}=(u,v)$ be the projection of the velocity vector onto the horizontal plane. We introduce a normal vector $\boldsymbol{n}=(\xi,\eta)/\sqrt{\xi^2+\eta^2}$ and a tangential vector $\boldsymbol{\tau}=(-\eta,\xi)/\sqrt{\xi^2+\eta^2}$ to the discontinuity front Γ_t — the section of the surface Γ by the plane $t=\text{const}$. Next, we determine the tangential and normal velocity vector components ($u_\tau=\boldsymbol{u}\cdot\boldsymbol{\tau}$ and $u_n=\boldsymbol{u}\cdot\boldsymbol{n}$, respectively) on Γ_t and the velocity of motion of the discontinuity front Γ_t ($D_n=-\tau/\sqrt{\xi^2+\eta^2}$). We use square brackets to denote the jump of the function in passing through the discontinuity: $[f]=f_2-f_1$, where f_2 and f_1 are the limiting values f on two sides of the discontinuity surface.

We consider solutions of system (2.1) in the class of functions having the following property: the functions u, v, and H, the vertical component of the vortex vector v_x-u_y, and their derivatives with respect to the variable λ are discontinuous piecewise-smooth functions which have a discontinuity of the first kind on the surface Γ. Let us find conditions to which the jump of the velocity vector on Γ should satisfy in this case. Let Ω be a region in the space (t,x,y,λ) located on both sides of Γ, and let φ be a smooth test function which vanishes on the boundary of the region Ω. Integration by parts yields the formula

$$\int\limits_{\Omega}(v_x-u_y)\varphi\,d\Omega=\int\limits_{\Omega}(-v\varphi_x+u\varphi_y)\,d\Omega+\int\limits_{\Gamma_\Delta}[u_\tau]\varphi\,d\Sigma. \tag{3.1}$$

Here Γ_Δ is part of the surface Γ which belongs to the region Ω. We consider a sequence of regions Ω_k ($k\to\infty$) such that the volume of the region tends to zero as $k\to\infty$ and $\bigcap\limits_k\Omega_k=\Gamma_\Delta$. Since the function v_x-u_y is locally integrable, the integrals in the region Ω_k in formula (3.1) tend to zero; then, by virtue of the arbitrariness of φ and Γ_Δ on Γ, the following relation should hold:

$$[u_\tau]=0. \tag{3.2}$$

Consequently, in the chosen class of functions, condition (3.2) is satisfied at the discontinuity front.

We elucidate what relations at a strong discontinuity can be derived from Eqs. (2.1) in the class of solutions indicated above. In the derivation of these relations, Eqs. (2.1) are integrated over similar regions Ω_k taking into account that, in the examined class of functions,

$$\int\limits_{\Omega_k}((v_x-u_y)u)_\lambda\varphi\,d\Omega\to 0,\qquad \int\limits_{\Omega_k}((v_x-u_y)v)_\lambda\varphi\,d\Omega\to 0 \tag{3.3}$$

as $k\to\infty$.

Similarly to that described above, using (3.3), we obtain the following relations at the discontinuity:

$$[H(u_n-D_n)]=0,\qquad [-D_n\boldsymbol{u}_\lambda+(|\boldsymbol{u}|^2/2)_\lambda\boldsymbol{n}]=0; \tag{3.4}$$

$$\Big[\int\limits_0^1 H(u_n-D_n)\boldsymbol{u}\,d\lambda+\frac{gh^2}{2}\,\boldsymbol{n}\Big]=0. \tag{3.5}$$

Multiplication of the second equation of system (3.4) by the tangential and normal vectors yields the following relations, respectively:

$$D_n[u_\tau]_\lambda=0,\qquad [(u_n-D_n)^2+u_\tau^2]_\lambda=0. \tag{3.6}$$

The first relation in the chosen class of functions holds by virtue of (3.2). In view of (3.2), the second relation in (3.6) can be written as

$$[(u_n-D_n)^2]_\lambda=0. \tag{3.7}$$

Scalar multiplication of Eq. (3.5) by the tangential vector $\boldsymbol{\tau}$ yields the equality

$$\int\limits_0^1 H(u_n - D_n)[u_\tau]\,d\lambda = 0,$$

which is satisfied automatically by virtue of (3.2). Multiplying Eqs. (3.5) scalarly by the normal vector $\boldsymbol{n}$ and performing some transformations, we obtain the following relation at the jump:

$$\Big[\int\limits_0^1 H(u_n - D_n)^2\,d\lambda + \frac{gh^2}{2}\Big] = 0. \tag{3.8}$$

Thus, the introduced definition of the class of solutions is compatible with the system of relations at the discontinuity. In this class, a number of relations following from (2.1) is satisfied automatically by virtue of the choice of the class, and, at the discontinuity front, it is necessary to satisfy only the first relation in (3.4) and relations (3.7) and (3.8).

From system (1.2), we also obtain the differential conservation law for the total energy of the fluid layer:

$$\Big(\int\limits_0^1 H\,\frac{u^2+v^2}{2}\,d\lambda + \frac{gh^2}{2}\Big)_t + \Big(\int\limits_0^1 Hu\Big(\frac{u^2+v^2}{2} + gh\Big)\,d\lambda\Big)_x + \Big(\int\limits_0^1 Hv\Big(\frac{u^2+v^2}{2} + gh\Big)\,d\lambda\Big)_y = 0.$$

At the front of the jump, let the inequality $[u_n - D_n] \neq 0$ be satisfied. The side of the jump front from which the fluid flows onto the front will be called the side before the jump, and the other side will be called the side behind the jump. Below, the quantities with subscript 2 correspond to the state behind the jump, and the normal to the front of the jump is directed toward the conditions behind the jump. We require that, at the jump, the condition of decreasing total energy of the fluid layer be satisfied:

$$\Big[\int\limits_0^1 H(u_n - D_n)\Big(\frac{(u_n - D_n)^2}{2} + gh\Big)\,d\lambda\Big] \leqslant 0 \tag{3.9}$$

[the equality sign in (3.9) corresponds to energy conservation in passing through the jump]. Using relations (3.4) and (3.7), this condition can be written as

$$\Big[\frac{(u_n - D_n)^2}{2} + gh\Big]\int\limits_0^1 H(u_n - D_n)\,d\lambda \leqslant 0.$$

Since, by virtue of the choice of the direction of the normal, $u_n - D_n > 0$, the energy decrease condition reduces to the inequality

$$\Big[\frac{(u_n - D_n)^2}{2} + gh\Big] = \Big(\frac{(u_{n2} - D_n)^2}{2} + gh_2\Big) - \Big(\frac{(u_{n1} - D_n)^2}{2} + gh_1\Big) \leqslant 0. \tag{3.10}$$

Thus, we obtained the system of relations at the discontinuity [Eqs. (3.4), (3.7), and (3.8)] and the stability condition for the discontinuity (3.10).

4. Definiteness Property of the System of Relations at the Jump. We prove that, if the velocity of the discontinuity surface moving along the normal D_n and the oncoming-flow parameters on one side of the discontinuity are known, the system of relations at the discontinuity allows one to determine the flow parameters on the other side of the discontinuity.

Using the notation $v_n = u_n - D_n$, we write relations (3.4), (3.7), (3.8), and (3.10) as

$$[Hv_n] = 0; \tag{4.1a}$$

$$\Big[\int\limits_0^1 Hv_n^2\,d\lambda + \frac{gh^2}{2}\Big] = 0; \tag{4.1b}$$

$$[v_n^2]_\lambda = 0; \tag{4.1c}$$

$$[v_n^2/2 + gh] \leqslant 0. \tag{4.1d}$$

Let the parameters H_1 and v_{n1} be known. To prove the formulated statement, it is sufficient to show that relations (4.1) allow one to uniquely determine the parameters behind the jump H_2 and v_{n2}.

Integration of relation (4.1c) over the variable λ yields the equality

$$v_{n2}^2 = v_{n1}^2 - K,$$

where K is an unknown quantity independent of λ. Using this equality and relation (4.1a), we obtain

$$v_{n2} = \pm\sqrt{v_{n1}^2 - K}, \qquad H_2 = \pm H_1 v_{n1}/\sqrt{v_{n1}^2 - K}\,. \tag{4.2}$$

According to the choice of the normal (see Sec. 3), $v_{n1} > 0$ and $v_{n2} > 0$; therefore, in (4.2) we choose the plus sign. Substitution of the obtained quantities into relation (4.1b) yields the equation

$$\int_0^1 H_1 v_{n1}\sqrt{v_{n1}^2 - K}\,d\lambda + \frac{g}{2}\Big(\int_0^1 \frac{H_1 v_{n1}}{\sqrt{v_{n1}^2 - K}}\,d\lambda\Big)^2 = \int_0^1 H_1 v_{n1}^2\,d\lambda + \frac{g}{2}\Big(\int_0^1 H_1\,d\lambda\Big)^2$$

for determining the unknown K. Once K is found, the flow parameters behind the jump are defined by the formulas given above (4.2).

It is convenient to introduce a function $F(K)$ defined for $K < K_* \equiv \min\limits_{\lambda} v_{n1}^2(\lambda)$ by means of the formula

$$F(K) = \int_0^1 H_1 v_{n1}\sqrt{v_{n1}^2 - K}\,d\lambda + \frac{g}{2}\Big(\int_0^1 \frac{H_1 v_{n1}}{\sqrt{v_{n1}^2 - K}}\,d\lambda\Big)^2,$$

and rearrange the equation for K to the form

$$F(K) - F(0) = 0. \tag{4.3}$$

According to condition (4.1d), it is required to find the root of Eq. (4.3) that satisfies the inequality

$$G(K) = g(h_2 - h_1) + \frac{v_{n2}^2 - v_{n1}^2}{2} = g\int_0^1 \frac{H_1 v_{n1}}{\sqrt{v_{n1}^2 - K}}\,d\lambda - g\int_0^1 H_1\,d\lambda - \frac{K_1}{2} \leqslant 0.$$

The derivative of the function $F(K)$ is calculated as

$$F'(K) = -\frac{\omega(K)}{2}\int_0^1 \frac{H_1 v_{n1}}{\sqrt{v_{n1}^2 - K}}\,d\lambda = -\frac{1}{2}\,h_2(K)\,\omega(K) = h_2(K)\,G'(K), \tag{4.4}$$

where

$$\omega(K) = 1 - g\int_0^1 \frac{H_1 v_{n1}}{(v_{n1}^2 - K)^{3/2}}\,d\lambda.$$

We note that

$$G''(K) = -\frac{1}{2}\,\omega'(K) = \frac{3g}{4}\int_0^1 \frac{H_1 v_{n1}}{(v_{n1}^2 - K)^{5/2}}\,d\lambda > 0$$

and, hence, the function $G(K)$ is convex downward. We assume that K_1 is a root of Eq. (4.3) that satisfies the inequality $G(K_1) \leqslant 0$. We show that $G(K) < 0$ in the interval $(0, K_1)$. Indeed, since $F(K_1) = F(0)$, in the indicated interval there exists a point K_0 at which $F'(K_0) = 0$. According to (4.4), at this point, $G'(K_0) = 0$, and, by virtue of the inequality $G''(K) > 0$, this point is unique. At the point K_0, the function $G(K)$ reaches the single negative minimum, and its maximum value is reached at the ends of the interval $K = 0$ and $K = K_*$. Since $G(0) = 0$ and $G(K_1) \leqslant 0$, we have $G(K) < 0$ in the interval $(0, K_1)$. Let us show that the root of Eq. (4.3) satisfies the inequality $K_1 > 0$. Relations (4.4) imply the equalities

$$0 = F(K_1) - F(0) = \int_0^{K_1} F'(K)\,dK = \int_0^{K_1} h_2(K)\,G'(K)\,dK = h_2(K_1)\,G(K_1) - \int_0^{K_1} G(K)\,h_2'(K)\,dK.$$

Since $h_2'(K) > 0$, $h_2(K) > 0$, and $G(K) < 0$, it is easy see that the last equality is satisfied only for $K_1 > 0$. Then, the function $F(K)$ decreases in the interval $(0, K_0)$ and increases in the interval (K_0, K_*), and the point K_0 is the point of local minimum. As a result, it is established that, if the root K_1 of Eq. (4.3) satisfies the inequality $G(K_1) \leqslant 0$, then, $K_1 \in (0, K_*]$. We note that the condition $K_1 > 0$ leads to the inequality $G'(0) < 0$. This implies that $\omega(0) > 0$ and, in addition, the inequalities $\omega(K_1) < 0$, $F'(0) < 0$, and $F'(K_1) > 0$ are satisfied. Using relations (4.3), the last two inequalities can be written in equivalent form

$$1 - g\int_0^1 \frac{H_1}{v_{n1}^2}\,d\lambda > 0, \qquad 1 - g\int_0^1 \frac{H_2}{v_{n2}^2}\,d\lambda < 0. \tag{4.5}$$

According to [8], the equation

$$1 - g\int_0^1 \frac{H}{(u_n - D_n)^2}\,d\lambda = 0$$

defines the velocity D_n (velocity of motion of the characteristic surface along the normal $\boldsymbol{n}$). Therefore, inequalities (4.5) imply that, before the front of the jump, the flow is supercritical, and behind the front of the jump, it is subcritical. The root $K_1 \in (0, K_*]$ of Eqs. (4.3) exists only if

$$F(K_* - 0) \geqslant F(0).$$

For $F(K_* - 0) < F(0)$, Eq. (4.3) does not have roots in the interval $(0, K_*)$. In this case, it is possible to construct solutions with zones of return (relative to the jump) flow behind the jump. The flow behind the jump consists of a layer of fluid particles that passed through the jump [the flow parameters in it are determined by the formulas (4.2) at $K = K_*$] and a layer of fluid particles moving only behind the discontinuity. In this case, the relations at the discontinuity can be used only to determine the thickness of the zone of return flow. Similar configurations for plane–parallel flows were studied in [3, 5].

This work was supported by the Russian Foundation for Basic Research (Grant No. 07-01-00609), Program of the President of the Russian Federation on the State Support of Leading Scientific Schools (Grant No. NSh-5245.2006.1), and INTAS Foundation (Grant No. 05-109-4107).

REFERENCES

1. V. M. Teshukov, "Spatial stationary long waves in shear flows," *J. Appl. Mech. Tech. Phys.*,**45**, No. 2, 172–180 (2004).
2. V. M. Teshukov, "Spatial simple waves on a shear flow," *J. Appl. Mech. Tech. Phys.*, **43**, No. 5, 661–670 (2002).
3. V. M. Teshukov, "Hydraulic jump in the shear flow of an ideal incompressible fluid," *J. Appl. Mech. Tech. Phys.*, **36**, No. 1, 10–18 (1995).
4. V. M. Teshukov, "Hydraulic jump in shear flow of a barotropic fluid," *J. Appl. Mech. Tech. Phys.*, **37**, No. 5, 675–682 (1996).
5. A. A. Chesnokov, "Interaction of shear flows of an ideal incompressible fluid in a channel," *J. Appl. Mech. Tech. Phys.*, **47**, No. 6, 800–811 (2006).
6. V. N. Monakhov and N. V. Khusnutdinova, "Conjugation of the channel and filtration flows of a viscous incompressible fluid," *J. Appl. Mech. Tech. Phys.*, **36**, No. 1, 86–90 (1995).
7. V. N. Monakhov and N. V. Khusnutdinova, "Conjugation of the main mathematical models of filtration of two-phase fluids," *Mat. Model.*, **14**, No. 10, 109–115 (2002).
8. V. M. Teshukov, "Hyperbolicity of long-wave equations," *Dokl. Akad. Nauk SSSR*, **284**, No. 3, 555–559 (1985).